

The Domino Problem of the Hyperbolic Plane Is Undecidable

Maurice Margenstern,
 Université Paul Verlaine – Metz,
 LITA, EA 3097, IUT de Metz,
 Île du Saulcy,
 57045 METZ Cédex, FRANCE,
e-mail: margens@univ-metz.fr

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Abstract

In this paper, we prove that the general tiling problem of the hyperbolic plane is undecidable by proving a slightly stronger version using only a regular polygon as the basic shape of the tiles. The problem was raised by a paper of Raphael Robinson in 1971, in his famous simplified proof that the general tiling problem is undecidable for the Euclidean plane, initially proved by Robert Berger in 1966.

1 Introduction

The question, whether it is possible to tile the plane with copies of a fixed set of tiles was raised by Wang, [24] in the late 50's of the previous century. Wang solved the *origin-constrained* problem which consists in fixing an initial tile in the above finite set of tiles. Indeed, fixing one tile is enough to entail the undecidability of the problem. The general case, later called the **general tiling problem** in this paper, *GTP* in short, without condition, in particular with no fixed initial tile, was proved undecidable by Berger in 1966, [1]. Both Wang's and Berger's proofs deal with the problem in the Euclidean plane. In 1971, Robinson found an alternative, simpler proof of the undecidability of the general problem in the Euclidean plane, see [22]. In this 1971 paper, he raises the question of the general problem for the hyperbolic plane. Seven years later, in 1978, he proved that in the hyperbolic plane, the origin-constrained problem is undecidable, see [23]. Up to now, *GTP* remained open.

In this paper, we give a synthetic presentation of the techniques contained on several technical papers deposited on *arXiv*, see [13, 18], and on the web site of the author, in particular [15].

In the second section, we remind a few features of all proofs of this problem. Then, we turn to the hyperbolic case, assuming that the reader is familiar with hyperbolic geometry, at least with its popular models: Poincaré's disc or half-plane. We refer the reader to [20] and to [10] for preliminaries and other bibliographical references.

In the third section, we sketchily present the frame of the construction. In the fourth section, we present a needed interlude, a parenthesis on brackets, which is a basic ingredient of the proof. In the fifth section, we lift up this line construction to a planar Euclidean one. In the sixth section, we show how to implement the Euclidean construction in the hyperbolic plane, using the specific properties indicated in the third section. In the seventh section, we complete the proof of the main result:

Theorem 1 *The domino problem of the hyperbolic plane is undecidable.*

From theorem 1, we immediately conclude that GTP is undecidable in the hyperbolic plane.

In the eighth section, we give several corollaries of the construction and we conclude in two directions. The first one wonders whether it is possible to simplify the construction. The second direction tries to see what might be learned from the construction leading to theorem 1.

Before turning to section 2, let us remark that an alternative proof of GTP is claimed by Jarkko Kari, see [6, 7]. His proof is completely different of this one. It is completely combinatoric and it makes use of a non-effective argument. Here, we have no room to discuss this latter point.

2 The general strategy

In the proofs of GTP in the Euclidean plane by Berger and Robinson, there is an assumption which is implicit and was, most probably, considered as obvious at that time.

Consider a finite set S of **prototiles**. We call **solution** of the tiling of the plane by S a partition \mathcal{P} such that the closure of any element of \mathcal{P} is a copy of an element of S . We notice that this definition contains the traditional condition on matching signs in the case when the elements of S possess signs.

Note that GTP can be formalized as follows:

$$\forall S \quad (\exists \mathcal{P} \text{ sol}(\mathcal{P}, S)) \vee \neg(\exists \mathcal{P} \text{ sol}(\mathcal{P}, S)),$$

where \vee is interpreted in a constructive way: there is an algorithm which, applied to S provides us with 'yes' if there is a solution and 'no' if there is none.

The origin-constrained problem can be formalized in a similar way by:

$$\forall (S, a) \quad (\exists \mathcal{P} \text{ sol}(\mathcal{P}, S, a)) \vee \neg(\exists \mathcal{P} \text{ sol}(\mathcal{P}, S, a)),$$

where $a \in S$, with the same algorithmic interpretation of \vee .

Now, note that if we have a solution of GTP , we also have a solution of the origin-constrained problem, with the facility that we may choose the first tile. However, to prove that GTP has no solution, we have to prove that,

whatever the initial tile, the corresponding origin-constrained problem also has no solution.

However, Berger's and Robinson's proofs consider that we start the construction with a special tile, called the **origin**. There is no contradiction with what we have just said, because they force the tiling to have a dense subset of origins. In the construction, the origins start the simulation of the space-time diagram of the computation of a Turing machine M . All origins compute the same machine M which can be assumed to start from an empty tape. The origins define infinitely many domains of computation of infinitely many sizes. If the machine does not halt, starting from an origin, it is possible to tile the plane. If the machine halts, whatever the initial tile, we nearby find an origin and, from this one, we shall eventually fall into a domain which contains the halting of the machine: at this point, it is easy to prevent the tiling to go on.

The present construction aims at the same goal.

3 The general frame: the tiling $\{7, 3\}$ and its mantilla

Our construction takes place in a particular tiling of the hyperbolic plane: the tessellation $\{7, 3\}$ which we call the **ternary heptagrid**, simply **heptagrid**, for short, see [2, 11, 20]. It is generated by the regular heptagon with vertex angle $\frac{2\pi}{3}$ by reflection in its sides and, recursively, by reflection of the images in their sides. The background of figure 3 gives an illustration of this tiling in the Poincaré disc.

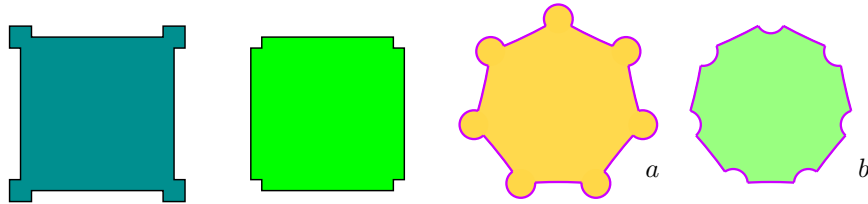


Figure 1 *On the left: Robinson's basic tiles for the undecidability of the tiling problem in the Euclidean case. On the right: the tiles a and b are a 'literal' translation of Robinson's basic tiles to the situation of the ternary heptagrid.*

We consider a special tiling based on the tessellation $\{7, 3\}$ which is motivated by the following consideration. In figure 1, the left-hand side indicates the basic shapes of the tiles devised by Robinson in the construction of the tiling used in his proof of the undecidability of GTP . The right-hand side of the figure gives the 'literal' translation of these tiles for the heptagrid. It is not difficult to see that it is not possible to tile the hyperbolic plane with the tiles a and b . However, a slight modification of the tile b , see the tile c of figure 2, leads to the solution.

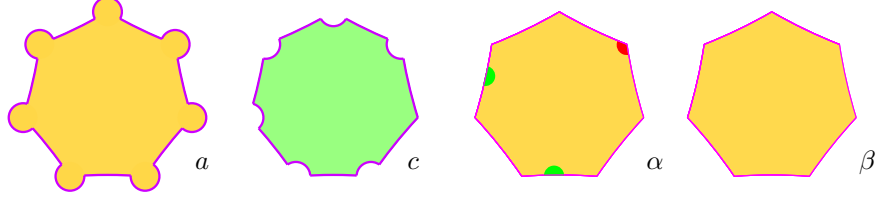


Figure 2 On the left: change in the tiles à la Robinson. On the right: their translation in pure Wang tiles.

On the right-hand side of figure 2, we have the translation of the tiles a and c into genuine à la Wang tiles. The pattern α corresponds to a and the pattern β corresponds to c .

3.1 The mantilla

In the ternary heptagrid, a **ball** of **radius** n around a tile T_0 is the set of tiles which are within distance n from T_0 which we call the **centre** of the ball. The **distance** of a tile T_0 to another T_1 is the number of tiles constituting the shortest path of adjacent tiles between T_0 and T_1 . We call **flower** a ball of radius 1.

Now, the tiles α and β of figure 2 can be assembled in flowers only: a tile α , which we call **centre**, require to be surrounded by tiles β only, which we call **petals**. This is obtained by numbering the edges of the tiles α from 1 to 7, see [14, 15]. Now, a petal belongs to three flowers at the same time by the very definition of the implementation. From this, there is a partial merging of the flowers. By definition, the resulting tiling is the **mantilla**.

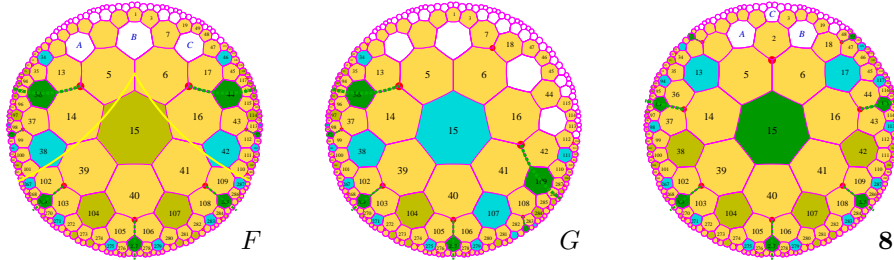


Figure 3 Splitting of the sectors defined by the flowers. From left to right: an F -sector, a G -sector and an 8 -sector.

It is not difficult to see that there can be several types of flowers, considering the number of red vertices for which the other end of an edge is a vertex of a centre. We refer the reader to [14] for the corresponding properties. Here, we simply take into consideration that we have three basic patterns of flowers, which we call F -, G - and 8 -flowers respectively. They are represented by figure 3.

The figure also represents the way which allows to algorithmically construct the mantilla. It consists in splitting the **sectors** generated by each kind of flowers in sub-sectors of the same kind and only them, which we call the **sons** of the flowers. From this, we easily devise a way to recursively define a tiling. The construction is deterministic below the flower, and it is non-deterministic when we proceed upwards. We do not make the notion of top and bottom more precise: it will be done later. The exact description of the splitting can be found in [14]. We simply remark that such a splitting is an application of the general method described in [20, 10], for instance.

Based on these considerations, we have the following result which is thoroughly proved in [14].

Lemma 1 *There is a set of 4 tiles of type α and 17 tiles of type β which allows to tile the hyperbolic plane as a matnilla. Moreover, there is an algorithm to perform such a construction.*

3.2 Trees of the mantilla

Note that the leftmost flower of figure 3, which represents an F -sector, also indicates a region delimited by continuous lines, yellow in coloured figures. These lines are **mid-point** lines, which pass through the mid-points of consecutive edges of heptagons of the heptagrid. As shown in [2, 11], they delimit a Fibonacci tree. Let us remind that a Fibonacci tree has two kinds of nodes: black ones and white ones. A black node has two sons, a black and a white one. A white node has three sons, a black and two white ones. In both cases, the black son is the leftmost son. The root of a Fibonacci tree is a white node. The tiles inside the tree which are cut by these mid-point rays are called the **borders** of the tree, while the set of tiles spanned by the Fibonacci tree is called the **area** of the tree.

Say that an F -son of a G -flower is a **seed** and the tree, rooted at a seed is called a **tree of the mantilla**. As the seeds are the candidates for the construction of a computing region, they play an important rôle. From figure 3,, we can easily define the **border** of a sector which is a ray crossing **8**-centres. See [14] for exact definitions.

Lemma 2 *The borders of a tree of the mantilla never meet the border of a sector.*

From lemma 2, as shown in [14], we easily obtain:

Lemma 3 *Consider two trees of the mantilla. Their borders never meet. Either their areas are disjoint or the area of one of them contains the area of the other.*

From this, we can order the trees of the mantilla by inclusion of their areas. It is clear that it is only a partial order. We are interested by the maximal elements of this order. We call them **threads**, see [14] for an exact definition. Threads are indexed by \mathbb{N} or \mathbb{Z} . We call **ultra-threads** the threads which are

indexed by \mathbb{Z} . When there are ultra-threads, two of them coincide, starting from a certain index. Note that the union of the areas of the trees which belong to an ultra-thread is the hyperbolic plane. There can be realizations of the mantilla with or without ultra-threads.

3.3 Isoclines

In [15], we have a new ingredient. We define the status of a tile as **black** or **white**, defining them by the usual rules of such nodes in a Fibonacci tree. Then, we have the following property.

Lemma 4 *It is possible to require that **8-centres** are always black tiles. When this is the case, a seed is always a black tile.*

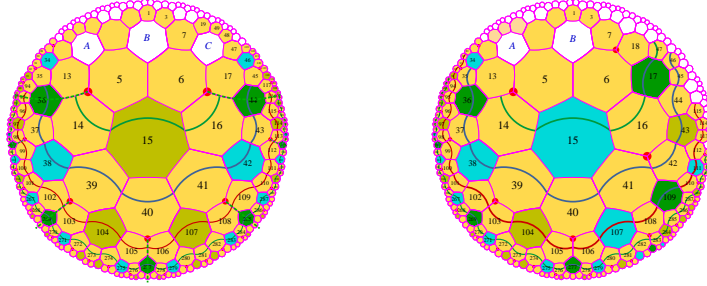


Figure 4 *The black tile property and the levels: On the left-hand side, a black F -centre; on the right-hand side, a black G_ℓ -centre. We can see the case of an **8-centre** on both figures.*

As shown in [15], we can define arcs as follows: in a white tile, the arc joins the mid-point of the sides which have a common vertex with the side shared by the father. In a black tile, the arc joins the mid-point of the sides shared by the father and the side shared by the uncle, which is on the left-hand side of the father. Joining arcs, we get paths. The maximal paths are called **isoclines**. They are illustrated on figure 4. An isocline is infinite and it splits the hyperbolic plane into two infinite parts. The isoclines from the different trees match, even when the areas are disjoint.

Lemma 5 *Let the root of a tree of the mantilla T be on the isocline 0. Then, there is a seed in the area of T on the isocline 5. If an **8-centre** A is on the isocline 0, starting from the isocline 4, there are seeds on all the levels. From the isocline 10 there are seeds at a distance at most 20 from A .*

We number the isocline from 0 to 19 and repeat this, periodically. This allows to give sense to **upwards** and **downwards** in the hyperbolic plane.

4 A parenthesis on brackets

We refer the reader to [15] for an exact definition. However, figure 5, below, illustrates the construction which now, we sketchily describe.

The generation 0 consists of points on a line which are regularly spaced. The points are labelled R , M , B , M , in this order, and the labelling is periodically repeated. An interval defined by an R and the next B , on its right-hand side, is called **active** and an interval defined by a B and the next R on its right-hand side is called **silent**. The generation 0 is said to be **blue**.

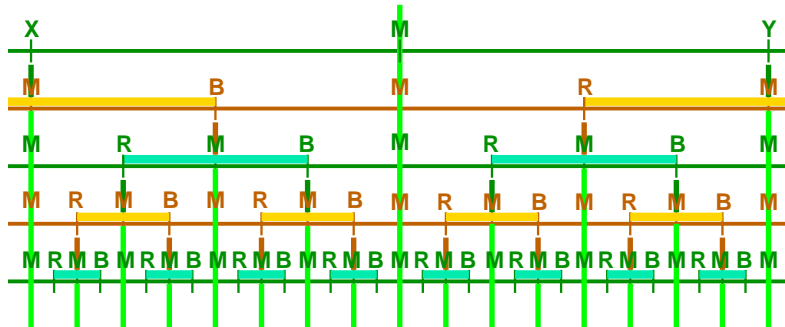


Figure 5 *The silent and active intervals with respect to mid-point lines. The light green vertical signals send the mid-point of the concerned interval to the next generation. The colours are chosen to be easily replaced by red or blue in an opposite way. The ends X and Y indicate that the figure can be used to study both active and silent intervals.*

Blue and red are said **opposite**. Assume that the generation n is defined. For the generation $n+1$, the points which we take into consideration are the points which are still labelled M when the generation n is completed. Then, we take at random an M which is the mid-point of an active interval of the generation n , and we label it, either R or B . Next, we define the active and silent intervals in the same way as for the generation 0. The active and silent intervals of the generation $n+1$ have a colour, opposite to that of the generation n .

When the process is achieved, we get an **infinite model**.

Deep results on the space of all these realizations are given by an accurate analysis to be found in [8]. The interested reader should have a look at this paper.

For our purpose, infinite models have interesting properties, see [15]. We cannot mention all of them here. We postpone some of them to the Euclidean implementation with triangles.

Cut an infinite model at some letter and remove all active intervals which contain this letter. What remains on the right-hand side of the letter is called a **semi-infinite model**.

It can be proved that in a semi-infinite model, any letter y is contained in at most finitely many active intervals, see [15].

5 The interwoven triangles

Now, we lift up the active intervals as *triangles* in the Euclidean plane. The triangles are isocles and their heights are supported by the same line, called the **axis**, see figure 6.

We also lift up silent intervals of the infinite model up to again isocles triangles with their heights on the axis. To distinguish them from the others, we call them **phantoms**. We shall speak of **trilaterals** for properties shared by both triangles and phantoms.

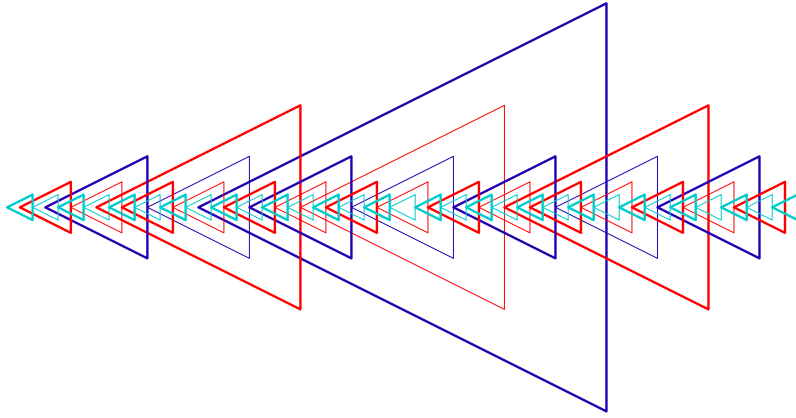


Figure 6 *An illustration of the interwoven triangles.*

We have very interesting properties for our purpose.

Lemma 6 *Triangles of the same colour do not meet no overlap: they are disjoint or embedded. Phantoms can be split into **towers** of embedded phantoms with the same mid-point and with alternating colours. Trilaterals can meet by a basis cutting the halves of the legs which contains the vertex.*

From the construction of the abstract brackets, we get a simple algorithm to construct the interwoven triangles.

The generation 0 is fixed by alternating triangles and phantoms. The mid-distance lines of the phantoms of generation 0 grow a horizontal green signal which crosses the legs of the phantom which they meet.

When the generation n is completed, a vertex of a triangle or a phantom is put on the intersection of the axis with the mid-distance line of a **triangle** of generation n . The legs grow until they meet a green signal, traveling on a horizontal line. If the trilateral is a triangle, the legs stop the green signal. If the trilateral is a phantom, the green signal crosses the legs. In both cases, the legs go on, until they meet a basis of their colour. They stop it and constitute a trilateral of the generation $n+1$. Now, the intersection of the basis of the

trilateral with the axis requires a vertex: of a triangle, if the basis belongs to a phantom, of a phantom, if the basis belongs to a triangle. The process is repeated endlessly.

The detection of the green signal and of the correct basis are facilitated by the following mechanism: the legs of red triangles emit horizontal signals which have a laterality. A right-hand side signal emits a right-hand side signal and a left-hand side leg emits a left-hand side signal. We do not provide a tile for the meeting of such signals inside a triangle. This allows to detect what corresponds to the free letters and which we call the **free rows** of a red triangle.

Lemma 7 *The interwoven triangles can be obtained by a tiling of the Euclidean plane which can be forced by a set of 190 tiles.*

In [15], we display the corresponding tiles which are in a square format, and we also describe them with the help of formulas taking into account the properties of lemma 6.

6 Implementing the interwoven triangles in the hyperbolic plane

The idea of the implementation in the hyperbolic plane is based on the following observation.

From lemma 5, we define the isoclines 0, 5, 10 and 15 to play the rôle of the rows in the Euclidean implementation. The trilaterals will be constructed on trees of the mantilla. A vertex will be a seed, and the legs are supported by the borders of the tree. The basis is defined by an isocline which cuts both borders of the tree.

As there are 6 seeds on the isocline 5 inside the tree defined by a seed at the isocline 0, there are 6 trilaterals of the generation 1 raised by a triangle of the generation 0. And so, contrarily to what happens in the Euclidean construction, we have several trilaterals of the same generation for the same set of isoclines crossed by the legs of these trilaterals.

Call **latitude** of a trilateral the set of isoclines which are crossed by its legs, vertex and basis being included.

It is not difficult to see that there will be infinitely many trilaterals within a given latitude. This requires to **synchronize** the choice of triangle or phantom when turing from one generation to the next one. Also, we can imagine that horizontal signals coming from different triangles of the same latitude and with different lateralities will meet. This will require to tune many details in order to maintain the guidelines of the algorithm, described in section 5.

The idea will be to synchronize the bases, the vertices and the signals on the mid-distance lines. We shall also have to see how the axis is implemented.

From what we have already seen, there is no more one axis, but a lot of them. In fact, what we called an axis can be materialized by a thread. AS most

threads are indexed by \mathbb{N} only, we have always the implementation of a semi-infinite model. Now, we shall manage the implementation in such a way that the semi-infinite models are simply different cuts of the same infinite model. The possibility of the realization of the infinite model in the case of an ultra-thread brings in no harm: it can be viewed as a cut at infinity.

6.1 The scent

By definition, we decide that all seeds which are on an isocline 0 are **active**, which means that they actually grow legs of a triangle of the generation 0. This is enough to guarantee that the set of active seeds is dense in the hyperbolic plane. Next, an active seed diffuses a **scent** inside its trilateral until the fifth isocline, starting from this seed, is reached. Seeds which receive the scent, and only them, become active. An active seed triggers the green signal when it reaches an isocline 5 or an isocline 15. By construction, the generation 0 is not determined by the meeting of a green signal. But the others are.

We can see that the scent process constructs a tree. The branches of the tree materilize the thread which implements the considered semi-infinite models. Note that the above synchronizatoin mechanism fixes things for spaces between triangles but also inside them.

6.2 Horizontals signals with a laterality

Due to the occurrene of several trilaterals within a given latitude, now we have to require that all triangles, both red and blue and blue 0, emit a signal along their legs. The signal will be called **upper** when it is emitted by the legs, but there is an exception: the vertices do not emit any horizontal signal. There is another exception: the corners of a phantom, as well as those of a triangle, also emit an upper signal. The colour and the laterality of this signal are those of the corner.

Now, in between two contiguous triangles of the same latitude, horizontal signals of the same colour but with a different laterality will meet. We have to allow such a meeting, which will be performed by an appropriate tile which we call a **join** tile. There is a join-tile for red and blue signals, as well as for the orange signals. The join-tile, see the pattern represented by figure 7, illustrates such a junction. On the left-hand side, we have the right-hand side signal and, on the right-hand side, we have the left-hand side one. We also require that an upper horizontal signal of a given laterality may cross a leg of a trialteral of the same colour, only if it has the same laterality as the leg. Note that the opposite junction cannot be obtained, as turing tiles is impossible in our setting. This impossibility is guaranteed by the existence of the isoclines and thir numbering.

Lemma 8 *An upper horizontal signal with a laterality cannot join two legs of the same tilateral.*

Proof: easy corollary of the rule about the meeting of legs with an upper horizontal signal.

Accordingly, the legs of a trilateral may only be joined by a horizontal signal which has no laterality.

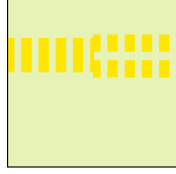


Figure 7 *The pattern of a join-tile.*

6.3 Synchronisation: the mechanisms

The first mechanism which we introduce to force the synchronization of different constructions is that all bases on a given isocline merge. This changes the tile for the corner, but this does not affect the algorithm of section 5.

The first principle forces the presence of various signals on the same isocline. We have to look at the consequences of such simultaneous presence in order to avoid contradictions which would ruin the construction.

As a first point, note that the upper signals allow to differentiate the various parts of a basis. If the upper horizontal signal is of the same colour as the basis, we are outside any trilateral whose basis lies on the same isocline. If not, we are inside. We say that the basis is **covered** if it is accompanied by an upper horizontal signal of its colour. Otherwise, we say that the basis is **open**.

This distinction is important. When a leg meets a basis: if it is the first half of a leg, *i.e.* between the vertex and the mid-point of the leg, it meets the basis without changing it. When the second half of a leg meets a basis, it crosses it if the colour is different. If it is the same colour, it crosses it only if the basis is covered. If it is open, the leg has met the basis with which it forms a corner. Indeed, from lemma 8, an upper horizontal signal cannot go from one leg of a trilateral to the other: there must be a triangle of the same colour in between. And so, an open basis does not cross the second half of a leg of a trilateral of the same colour: and so, for the second half of a leg, when the leg meets such a signal, this means that the expected basis is found.

Now, we have to look at the consequences of the synchronization on other signals: on the green signals and on the horizontal blue and red signals.

The problem is the following. Consider two triangles A and B of the same latitude. They belong to the same generation n and they have the same colour. We may assume that A is on the left-hand side of B . Let ι be the isocline of the mid-distance line of A and B . From the study of the abstract brackets, we know that there is a tower of phantoms inside A and inside B , with also ι as the mid-distance line of the phantoms of the tower. We also know that the same tower is repeated along ι between A and B , possibly several times. Accordingly, there is a green signal on ι inside A and inside B , and also between A and B .

A similar problem occurs with the horizontal signals which are emitted by the right-hand side leg of A , the vertex being excepted. Symmetrically, the left-hand side leg of B emits a horizontal signal of the same colour and on the same isoclines of the latitude of A . The join-tile of figure 7, replicated in appropriate colours, allows to connect together horizontal signals of opposite lateralities travelling between A and B in the appropriate directions and on the same isocline. From lemma 8, if a horizontal signal enters a phantom of its colour, we get into trouble for the meeting with the other leg of the phantom.

Both problems can be solved in a similar way.

The idea is to **avoid** the phantoms as it is not possible to cross them. The advantage is that the deviated signal does not disturb the construction inside the phantom. Also, if during the detour, the signal keeps its laterality, it behaves as if the phantom were not present. In particular, the join-tile can be used for the connection of the signal with the opposite one coming in front of it, from the other triangle.

Technically, the solution is the following. The legs of a triangle stop the green signal which meet them on the mid-distance line, as required by the algorithm of section 5. Now, at the mid-point of the leg, on the corresponding isocline ι , the leg triggers an **orange** signal σ of its laterality, outside the triangle, say A . When σ meets the first phantom P on its way, it does not cross it: on the other side of the leg of P , there is a green signal. Instead of this, σ climbs up over the left-hand side leg of P until it reaches the vertex. Then, it goes down along the right-hand side leg of P until it reaches the isocline σ . This is easy to recognize: it is the single tile such that there is a green signal on the other side of the leg. Also, during this travel on the first half of the legs of P , σ does not change its laterality. From this, lemma 8 also applies to σ , which rules out the occurrence of an orange signal on ι inside P . And so, σ does not disturb the construction inside P . By lemma 8, we can see that σ matches its opposite signal on ι with the join-tile.

It is not difficult to see that the solution applies to a horizontal blue signal. Indeed, the structure of inner trilaterals inside a phantom is the same as inside a triangle of the same generation. Accordingly, the notion of a free row also applies to phantoms. Now, for a blue-0 or blue phantom, there is a single free row: the mid-distance line. Accordingly, a horizontal blue signal meeting the leg of a blue-0 or blue phantom P on an isocline which is not the mid-distance line of P also meets the opposite signal coming from a blue-0 or blue triangle inside the phantom. The join-tile solves the problem if needed. For the blue signal which travels on the isocline ι of the mid-distance line of P , its behaviour is exactly that of an orange signal, which solves the problem.

We remain with the case of a red triangle. This time, ι still denotes the mid-distance line of the triangles A and B , and the right-hand side leg of A emits right-hand side horizontal red signals on all the isoclines of the latitude of A , the line of the vertex of A being excepted. We have the same situation as with a blue signal if the signal arrives on an isocline which does not correspond to a free row of the red phantom P which is first met on the way. If the signal arrives on an isocline of a free row of P , the idea is to collect all the red signals

travelling within the latitude of P on the isocline of a free row in a single signal, as there are a lot of free rows in a red trilateral. This single signal has the form of a red signal with a laterality: that of all the red signals arriving to P . It climbs up over the legs of P , this time from the vertex to the basis and conversely. Also, the laterality of the signal is not changed and, when it goes down along the other leg of P , the signal sends a copy of itself, outside P , on each isocline of a free row. In this way, the red signals which arrive to P and which depart from it on an isocline of a free row of P constitute a comb: on one side, the comb gathers the signals, and on the other side, it dispatches them. In this way, the avoiding is obtained without perturbing what happens inside P , and also without disturbing what must be outside P . The signal passes as if P were not present. We just remark that as the laterality is not changed, lemma 8 also applies here. As a consequence, the same phenomenon may happen inside P , but only within the latitudes of the triangles which P contains. As the mid-distance line of these triangles are different from that of P , the just described phenomenon occurs for the inner phantoms inside P whose mid-distance line is that of P . This is conformal with the requirement that what happens inside P must not be disturbed.

Now, we can conclude that the tiling forces the construction of trilaterals generation after generation, as indicated by the algorithm of section 5.

7 Completing the proof

7.1 The computing areas

The **active** seeds were defined in sub-section 6.1. They allow to define the trilaterals in the hyperbolic plane.

Now, we ignore the blue-0 and the blue triangles, the phantoms of any colour as well as the parts of the bases of red triangles which are covered. Accordingly, we focus our attention on the red triangles only.

We have already mentioned that lemma 8 applied to horizontal red signals allows us to detect the free rows inside the red triangles. We shall agree that a special signal, the yellow one, without laterality, will mark the free rows of the red triangles. In fact, the set of tiles which implement the interwoven triangles in the Euclidean plane, see [13, 15], also implement this detection and the installation of the yellow signal on the free rows.

The free rows of the red triangles constitute the horizontals of the grid which we construct in order to simulate the space-time diagram of a Turing machine.

Now, we have to define the verticals of the grid to complete the simulation.

The verticals consist of rays which cross **8**-centres. Figure 8 illustrates how they are connected to the different possible cases of contact of the isocline of free row with the border of the tree.

The computing signal starts from a the seed. It travels on the free rows. Each time a vertical is met, which contains a symbol of the tape, the required instruction is performed. If the direction is not changed and the corresponding

border is not met, the signal goes on, on the same row. Otherwise, it goes down along the vertical until it meets the next free row. There, it looks at the expected vertical, going in the appropriate direction. Further details are dealt with in [15].

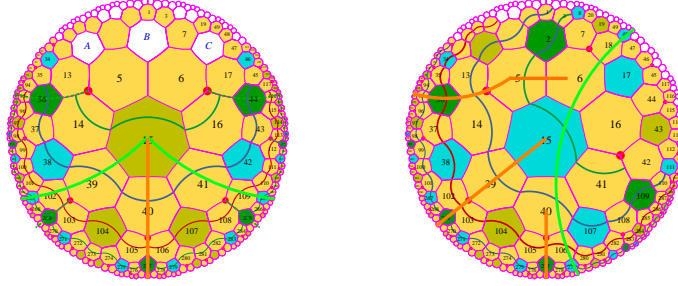


Figure 8 *The perpendicular starting from a point of the border of a triangle which represents a square of the Turing tape.*

On the left-hand side: the case of the vertex. On the right-hand side, the three other cases for the right-hand side border are displayed on the same figure.

Note that in the case of the butterfly model, see [13, 15], the mechanism of the orange signal forces the green signal to run over the whole isocline which is the mid-point of the latitude which contains no triangle. Indeed, the laterality constraints of the tiles for the crossing legs of trilaterals prevent an orange signal to run at infinity.

7.2 The tiles

Within the frame of this paper, it is not possible to exhaustively describe the tiles needed for the construction which we described. In the reports [15] and in [13], we give a precise account of the tiles.

Here, we just indicate how to construct the tiles, at the same time giving a way to describe all the tiles and to count them.

The finite set of tiles which we need to prove theorem 1, consists of two parts. First, we have the set of **prototiles** which forces the construction of the mantilla as well as the isoclines, the activation of seeds through the isoclines 0 and the spreading of the scent, and then the construction of the interwoven triangles, including the detection of the free rows in the red triangles.

The second set consists of **meta-tiles** which, in fact, are *variables* for tiles, as the meta-tiles convey the signals directly connected with the simulation of a given Turing machine. In the actual construction, the meta-tiles replace a part of the prototiles: they replace all the prototiles which are placed on an element of the computation: either the tiles which convey the computing signal, or those which convey the evolution of each square of the Turing tape. But the replacement is not systematic: depending on the simulated machine, the same free row may hold a computing signal in a precise interval and no computing signal outside this interval.

In each set, the tiles are constituted of a tile (α) or (β) on which we superpose several signals. We distinguish between horizontal and vertical signals. The horizontal signals can be viewed as **channels** which run along the path of the isocline inside the tile. We know that each tile has a top part and a bottom one. By definition, the top is determined by the side which is shared by the father of the tile. We define a **local numbering** by numbering the sides of a tile from 1 up to 7, with the number 1 given to the side shared with the father, and the other numbers in increasing order while counter-clockwise turing around the tile. In the local numbering, the isocline goes from the side 2 to the side 7 in a white tile and it goes from the side 3 to the side 7 in a black one. For upper signals; the channel is above the isoclines. A basis runs below the isocline. The green and orange signals run on the same channel which is above the isocline and below the channel for a horizontal red or blue signal.

For vertical signals, we have two main kinds of verticals: the legs of a trilateral and the verticals for the computing grid. The signals of the legs always cross black tiles and they are split into two categories: right-hand side signals which go from the side 2 to the side 6 and left-hand side ones which go from the side 1 to the side 4. From this, we notice that a tile of a leg always knows its laterality and, also, which of its sides are inside the trilateral and which are outside it. The vertical signals for the grid go through **8-centres**. In an **8-centre**, the signal goes from the side 2 to the side 5 in terms of the local numbering. In terms of the numbering of the mantilla, the signal goes from the side 7 to the side $\bar{4}$. Next, in the petal $1\bar{4}7\circ$, it goes from the side 1 to the side 4 in terms of the local numbering. And then, in the petal $2\circ 77$, it goes from the side 1 to the side 6, still with respect to the local numbering.

We just remark that we have three colours for the trilaterals: blue-0 for the generation 0, blue for the even generations and red for the odd ones. Legs of tri-angles are represented by a **thick** signal while legs of phantoms are represented by a **thin** one. First and second halves of legs are also distinguished. In blue-0 and blue trilaterals, the first half is dark and the second half is light. In red trilaterals, we use the opposite convention: the first half is light and the second one is dark. The tile of figure 7 defines the general patterns to indicate the laterality of a signal. These patterns are given the colour of the corresponding signal.

Then, the most intricate task comes: the superposition of all these kinds of signals. This can be precisely described and counted, as performed in [15, 13].

With this, we completed the proof of theorem 1.

8 A few corollaries

The construction leading to the proof of theorem 1 allows to get a few results in the same line of problems.

As indicated in [3, 4], there is a connection between *GTP* and the **Heesch number** of a tiling. This number is defined as the maximum number of **coronas** of a disc which can be formed with the tiles of a given set of tiles, see [9] for

more information. As indicated in [4], and as our construction fits in the case of domino tilings, we have the following corollary of theorem 1.

Theorem 2 *There is no computable function which bounds the Heesch number for the tilings of the hyperbolic plane.*

The construction of [12, 14] gives the following result, see [16, 19].

Theorem 3 *The finite tiling problem is undecidable for the hyperbolic plane.*

Combined with the construction proving theorem 3 and a result of [20], the construction of the present paper allows us to establish the following result, see [17].

Theorem 4 *The periodic tiling problem is undecidable for the hyperbolic plane, also in its domino version.*

In this statement, **periodic** means that there is a shift which leaves the tiling globally invariant.

At last, in another direction, we may apply the arguments of Hanf and Myers, see [5, 21], and prove the following result, see [15].

Theorem 5 *There is a finite set of tiles such that it generates only non-recursive tilings of the hyperbolic plane.*

9 Conclusion

The first consequence is that, according to the estimations of [15, 18], we need a huge number of tiles. Taking into account the changes introduced in [18], a new counting indicates that we need 23,323 tiles for the prototiles and 6,541 additional ones for the meta-tiles, this will precisely be presented in a forthcoming paper.

Of course, we may wonder whether the number of tiles can be reduced. This might be possible by a small tuning of the present signals. As an example, we could forbid yellow rows on the mid-distance line of a red triangle. But the advantage would not be important enough. Now, an attentive look at the tables, suggested in sub-section 7.2, indicates that the reason of the big number of tiles lies in lemma 5. A consequence of the lemma is the important number of passive tiles connected with the isoclines which do not bear the construction signals. And so, to get a significant reduction of the number of tiles means to find a new, simpler setting. Professor Goodman-Strauss advised me to do so. In fact, recently, I realized that this is possible. I have not the room, here, to explain the idea. It is interesting to notice that this simplification allows to implement a similar implementation of the interwoven triangles in both the ternary heptagrid and in the pentagrid.

The second consequence which could be derived from the construction lies on a more abstract level. Let us look at the lifting of the abstract brackets to the

interwoven triangles. At first glance, this seems to be a Euclidean construction. In the very paper, a whole section is devoted to the Euclidean implementation of the interwoven triangles. And in the next section, we still transfer this construction to the hyperbolic plane. It seems to me that the fact that this transfer is possible has an important meaning. From my humble point of view, it means that a construction which looks like purely Euclidean has indeed a purely combinatoric character. It belongs to absolute geometry and it mainly requires Archimedes' axiom. Note that absolute geometry itself has no pure model. A realization is necessarily either Euclidean or hyperbolic. We suggest to conclude that, probably, the extent of absolute geometry is somehow under-estimated.

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